Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Professor F. Morley of Johns Hopkins University applied the geometry of the complex variable to obtain "Orthocentric properties of the plane n-line." He announced theorems relating to a general system of straight lines.

Four other papers on the programme and read by title were of geometrical character; they were by Dr. Snyder of Cornell, Professor Field of Carthage College, Dr. Keyser of Columbia University, Professor Emch of Colorado University.

On Wednesday afternoon were presented several papers not on the printed programme, also (by title) papers by Dr. Eisenhart of Princeton, Professor Roe of Syracuse University, Dr. Epsteen of Philadelphia, Mr. Ford of the University of Michigan, and finally the following three papers:

"The bilinear point-line connex in space; an extension of Clebsch's connex," by Dr. E. Kasner of Columbia University; "Multiple points of Lissajous's curves in two and three dimensions," by Mr. E. A. Hook, Milwaukee; "Null systems in space of five dimensions," by Professor John Eiesland, Theil College.

A large number of the members lunched together each day at Evanston's one hotel, and dined together in one of the College buildings Tuesday evening, afterwards visiting the University observatory. On Thursday, the day after the Society adjourned, a number of members visisted the Chicago Steel Works and the University of Chicago, lunching together at the Quadrangle Club.

THE EXPRESSION OF THE nth POWER OF A NUMBER IN TERMS OF THE nth POWERS OF OTHER NUMBERS, n BEING ANY INTEGER; AND THE DEDUCTION OF SOME INTERESTING PROPERTIES OF PRIME NUMBERS.*

By J. W. NICHOLSON, A. M., LL. D., Professor of Mathematics in the Louisiana State University.

Any polynomial of m terms is equal to, and may be expressed under the form of:

+[The sum of its terms taken in sets of $m-1$]
−[" " " " " $m-2$]
+[" " " " " $m-3$]
−[" " " " " $m-4$]
. . . .

---

*This paper was read to the American Mathematical Society several years ago and in presenting it the author asked whether or not anything of the kind had ever before been discovered. The inquiry was referred to Dr. Artemas Martin who reported that the main formula, designated in this paper by (4), had been discovered and presented to the London Mathematical Society, but that the paper contained none of the transformations or deductions embodied in my discovery. With the hope that others will be interested in the matter and reach still more beautiful and important deductions, the paper as presented by me to the Society is now submitted to the public.

Examples.

1. $a_1+a_2+a_3=[(a_1+a_2)+(a_1+a_3)+(a_2+a_3)]-[(a_1)+(a_2)+(a_3)]\ldots.(1).$

2. $a_1+a_2+a_3+a_4=$

$$\left.\begin{array}{l} +[(a_1+a_2+a_3)+(a_1+a_2+a_4)+(a_1+a_3+a_4)+(a_2+a_3+a_4)] \\ -[(a_1+a_2)+(a_1+a_3)+(a_1+a_4)+(a_2+a_3)+(a_2+a_4)+(a_3+a_4)] \\ +[(a_1)+(a_2)+(a_3)+(a_4)] \end{array}\right\}\ldots.(2).$$

3. In general, making $s=a_1+a_2+\ldots.a_m$, we have

$$\left.\begin{array}{llr} s=[(s-a_1)+(s-a_2)+ & \quad.\quad.\quad. & (s-a_m)] \\ \quad-[s-a_1-a_2)+(s-a_1-a_3)+ & \quad.\quad. & (s-a_{m-1}-a_m] \\ \quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad. & \quad.\quad. & .\quad.\quad.\quad. \\ -(-1)^m[(a_1+a_2)+(a_1+a_3)+ & \quad.\quad.\quad. & (a_{m-1}+a_m)] \\ \quad(-1)^m[(a_1)+(a_2)+\ldots.(a_m)] & & \end{array}\right\}\ldots.(3).$$

That (3) is true may be shown thus:

Any term of s, as a_1, occurs in the first [ ] $m-1$ times; in the second [ ], $\frac{(m-1)(m-2)}{2}$ times; in the third, $\frac{(m-1)(m-2)(m-3)}{2.3}$ times; and so on to the last in which it occurs 1 time. Hence the sum of all the a_1's in the second member is

$$[(m-1)-\frac{(m-1)(m-2)}{2}+\ldots.(-1)^m]a_1=a_1. \qquad \text{Q. E. D.}$$

In all that follows, n is supposed to be a positive integer and less than m, unless otherwise stated.

An interesting and important property of polynomials arranged under the form of (3) is set forth in the following

Proposition: *The second member of (3) may be raised to the nth power by raising each of the parts within the ( ) to that power, ignoring the signs before the* [ ]. Thus:

$$\left.\begin{array}{llr} s^n=[(s-a_1)^n+(s-a_2)^n+ & \quad.\quad.\quad. & (s-a_m)^n] \\ \quad-[(s-a_1-a_2)^n+(s-a_1-a_3)^n+ & \quad. & (s-a_{m-1}-a_m)^n] \\ \quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad. & \quad.\quad. & .\quad.\quad.\quad. \\ -(-1)^m[(a_1+a_2)^n+(a_1+a_3)^n+ & \quad.\quad. & (a_{m-1}+a_m)^n] \\ \quad(-1)^m[(a_1)^m+(a_2)^n+\ldots.(a_m)^n] & & \end{array}\right\}\ldots.(4).$$

Proof. Let us suppose s^n, and each of the terms of the second member, to be expanded. Let $G=m'a_1{}^pa_2{}^q\ldots.a_k{}^z$ be any particular term of s^n, where $p+q+\ldots.z=n$, and k any integer from 1 to m. Now G occurs $m-k$ times in the first [ ], $\frac{(m-k)(m-k-1)}{2}$ times in the second [ ], $\frac{(m-k)(m-k-1(m-k-2)}{2.3}$ times in the third [ ], and so on. Hence the sum of all the G's in the second member is

$$[m-k-\frac{(m-k)(m-k-1)}{2}+\ldots.(-1)^m]G=G, \qquad \text{Q. E. D.}$$

EXAMPLES.

1. Making $a_1=2$, $a_2=3$, $a_m=a_3=6$, whence $s=11$, and we have

$$11^n=9^n+8^n+5^n-6^n-3^n-2^n\ldots.(5),$$

where $n=2$ or 1.

2. Making $a_1=2$, $a_2=3$, $a_3=6$, $a_m=a_4=12$, we have

$$23^n=21^n+20^n+17^n+11^n-18^n-15^n-14^n-9^n-8^n-5^n+12^n+6^n+3^n+2^n\ldots.(6),$$

where $n=3$, 2 or 1.

3. Making $a_1=1$, $a_2=3$, $a_3=7$, $a_4=15$, $a_m=a_5=31$, we have

$$\begin{aligned}57^n=56^n&+54^n-53^n+50^n-49^n-47^n+46^n+42^n-41^n-39^n+38^n-35^n+34^n+32^n\\&-31^n+26^n-25^n-23^n+22^n-19^n+18^n+16^n-15^n-11^n+10^n+8^n-7^n+4^n\\&-3^n-1^n\ldots.(7).\end{aligned}$$

Here $n=4$, 3, 2 or 1.

Some additional properties of an interesting character are presented in the following corollaries to the preceding proposition.

COR. I. If $s=2^m-1$, then s^m may be expressed in terms of the nth powers of all the integers from 2^m-2 to 1, inclusive.

EXAMPLES.

1. For $a_1=1$, $a_2=2$, $a_m=a_3=4$, $s=7=2^3-1$, we have

$$(2^3-1)^n+5^n-4^n+3^n-2^n-1^n\ldots.(8),$$

where $n=2$ or 1.

2. For $a_1=1$, $a_2=2$, $a_3=4$, $a_m=a_4=8$, $s=15=2^4-1$, we have

$$\begin{aligned}(2^4-1)^n=14^n&+13^n-12^n+11^n-10^n-9^n+8^n+7^n-6^n-5^n+4^n\\&-3^n+2^n+1^n\ldots.(9),\end{aligned}$$

where $n=3$, 2 or 1.

NOTE.—By (8) we observe that the sum of the last seven terms of $(9)=0$. Hence,

$$15^n = 14^n + 13^n - 12^n + 11^n - 10^n - 9^n + 8^n \ldots.(10),$$

where $n=2$ or 1.

3. In general, to find $(2^m-1)^n$ we make $a_1=1$, $a_2=2$, $a_3=2^2$, $a_4=2^3, \ldots a_m=2^{m-1}$, and obtain

$$(2^m-1)^n = (2^m-2)^n + (2^m-3)^n - (2^m-4)^n \ldots. - (-1)^m 3^n$$
$$+ (-1)^m 2^n + (-1)^m \ldots.(11),$$

where $n=m-1$, $m-2$, ...2, or 1.

COR. II. If $s=(m-1)a_2 + \left(\frac{(m-1)(m-2)}{2} + 1\right)a_1$, then s^n may be expressed in terms of the nth powers of $2^m-2-2(m-2)^2$ integers, or less.

This is done by making $a_3=a_2+a_1$, $a_4=a_3+a_1$, $a_5=a_4+a_1$, etc., in (4), and eliminating the equal terms having contrary signs.

EXAMPLES.

1. Make $m=3$, $a_3=a_2+a_1$ and (4) becomes

$$(2a_2+2a_1)^n = (2a_2+a_1)^n + (a_2+2a_1)^n - a_2^2 - (a_1)^2 \ldots.(12),$$

where $n=2$ or 1.

Thus, for $a_2=7$, $a_1=5$, and $n=2$, we have

$$24^2 = 19^2 + 17^2 - 7^2 - 5^2 \ldots.(13).$$

2. Make $m=4$, $a_4=a_3+a_1$, $a_3=a_2+a_1$, and we have

$$(3a_2+4a_1)^n = (3a_2+3a_1)^n + (2a_2+4a_1)^n - (a_2+3a_1)^n$$
$$- (2a_2+a_1)^n + a_2^n + a_1^n \ldots.(14),$$

where $n=3$, 2, or 1.

Thus, for $a_2=7$, $a_1=5$, $n=3$, we have

$$41^3 = 36^3 + 34^3 - 22^3 - 19^3 + 7^3 + 5^3 \ldots.(15).$$

3. Make $m=5$, $a_5=a_4+a_1$, $a_4=a_3+a_1$, $a_3=a_2+1$, and we have

$$(4a_2+7a_1)^n = (4a_2+6a_1)^n + (3a_2+7a_1)^n - (2a_2+6a_1)^n - (3a_2+3a_1)^n$$
$$- (2a_2+4a_1)^n + (2a_2+3a_1)^n + (a_2+4a_1)^n + (2a_2+a_1)^n - a_2^n - a_1^n \ldots(16),$$

where $n=4$, 3, 2, or 1.

4. Making $m=6$, $a_6=a_5+a_1$, $a_5=a_4+a_1$, etc., we have

$$(5a_2+11a_1)^n = (5a_2+10a_1)^n + (4a_2+11a_1)^n - (4a_2+6a_1)^n - (3a_2+10a_1)^n$$
$$- (3a_2+8a_1)^n + (3a_2+5a_1)^n + (2a_2+8a_1)^n + (3a_2+3a_1)^n$$

$$\begin{aligned}&+(2a_2+6a_1)^n-(a_2+5a_1)^n-(2a_2+3a_1)^n-(2a_2+a_1)^n\\&=a_2^n+a_1^n\ldots.(17),\end{aligned}$$

where $n=5$, 4, 3, 2, or 1.

Thus, making $a_2=5$, $a_1=2$, we have

$$\begin{aligned}47^n=45^n+42^n-35^n-32^n-31^n+26^n+25^n+2^n+5^n-12^n\\-15^n-16^n+21^n+22^n\ldots.(18),\end{aligned}$$

where $n=5$, 4, 3, 2, or 1.

Making $a_2=13$, $a_1=-5$, transposing, we have

$$24^n=22^n+21^n-15^n-13^n-12^n+10^n+5^n+4^n+3^n-1^n\ldots.(19),$$

where $n=5$, 3, or 1.

Making $a_2=8$, $a_1=-3$, transposing, we have

$$15^n=14^n+13^n-10^n-9^n+7^n-6^n+3^n+2^n+1^n\ldots.(20),$$

where $n=5$, 3, or 1.

COR. III. The root of each term in (4) may be increased by any number, as c, provided that the second member be increased by $-(-1)^m c^n$. Thus:

$$\begin{aligned}(s+c)^n=&+[(s+c-a_1)^n+(s+c-a_2)^n+\quad.\quad.\quad.\quad(s+c-a_m)^n]\\&-[(s+c-a_1-a_2)^n+(s+c-a_1-a_3)^n+\ldots.(s+c-a_{m-1}-a_m)^n]\\&.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad\ldots.(21).\\&-(-1)^m[(c+a_1+a_2)^n+(c+a_1+a_3)^n+\quad.\quad.\quad(c+a_{m-1}+a_m)^n]\\&(-1)^m[(c+a_1)^n+(c-a_2)^n+\quad.\quad.\quad.\quad.\quad.\quad(c+a_m)^n]\\&-(-1)^m c^n\end{aligned}$$

EXAMPLES.

1. Adding 8 to the roots of each term in (8) and we get (10).
2. Adding -5 to the roots of each term in (13) and we get

$$19^2=14^2+12^2+5^2-2^2\ldots.(22).$$

3. Adding -22 to the roots of each term in (18), we get

$$25^n=23^n+22^n-17^n-13^n-9^n+7^n+6^n+4^n+3^n-1^n\ldots.(23),$$

where $n=5$, 3, or 1.

4. Adding -7 to the roots of each term in (19), we get

$$17^n=15^n+14^n-8^n+7^n-5^n-4^n-2^n\ldots.(24),$$

where $n=5$, 3, or 1.

In (21), by making $a_1 = a_2 = \ldots . a_m$, we have

$$(c+ma_1)^n = m[c+(m-1)a_1]^n - \frac{m(m-1)}{2}[c+(m-2)a_1]^n + .. - (-1)^m c^n . (25).$$

By (25) we may express the nth power of $c+ma_1$ in terms of the nth powers of the terms of the arithmetical progression

$$c+(m-1)a_1, \quad c+(m-2)a_1, \quad c+(m-3)a_1, \text{ etc.,}$$

multiplied by the binomial coefficients, m, $\frac{m(m-1)}{2}$, etc., respectively. Thus:

Making $c=5$, $m=6$, $a_1=7$, we have

$$47^n = 6.40^n - 15.33^n + 20.26^n - 15.19^n + 6.12^n - 5^n \ldots . (26),$$

where $n=5$, 4, 3, 2, or 1.

Making $c=41$, $m=6$, $a_1=1$, we have

$$47^n = 6.46^n - 15.45^n + 20.44^n - 15.43^n + 6.42^n - 41^n \ldots . (27).$$

Making $c=-15$, $m=6$, $a_1=6$, we have

$$21^n = 6.15^n - 15.9^n + 20.3^n - 15.(-3)^n + 6(-9)^n - (-15)^n \ldots . (28).$$

Hence, $21^n = 5.15^n - 9.9^n + 5.3^n$, where $n=4$ or 2; and $21^n = 7.15^n - 21.9^n + 35.3^n$, where $n=5$, 3, or 1.

Making $c=0$ and $a_1=1$, we have

$$m^n = m(m-1)^m - \frac{m(m-1)}{2}(m-2)^n + \ldots . + (-1)^m m.1^n \ldots . (29).$$

Thus, for $m=6$, we have $6^n = 6.5^n - 15.4^n + 20.3^n - 15.2^n + 6$, where $n=5$, 4, 3, 2, or 1.

COR. IV. Multiplying (4) by 2^n, and then adding $c-s$ to the root of each term, we have

$$\begin{array}{ll}
(c+s)^n = & [(c+s-2a_1)^n + (c+s-2a_2)^n + \quad . \quad . \quad (c+s-2a_m)^n] \\
& -[(c+s-2a_1-2a_2)^n \quad . \quad . \quad . \quad (c+s-2a_{m-1}-2a_m)^n] \\
& \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \ldots . (30). \\
-(-1)^m & [(c-s+2a_1+2a_2)^n \quad . \quad . \quad . \quad [c-s+2a_{m-1}+2a_m)^n] \\
(-1)^m & [(c-s+2a_1)^n + (c-s+2a_2)^n + \quad . \quad . \quad (c-s+2a_m)^n] \\
-(-1)^m & (c-s)^n
\end{array}$$

EXAMPLES.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_m=a_4=5$, $c=7$, we have

$$18^n=16^n+14^n-10^n-2^n\text{, or }9^n=8^n+7^n-5^n-1^n\ldots.(31),$$

where $n=3$ or 1.

Making $a_1=1$, $a_2=2$, $a_3=3$, $a_4=5$, $a_5=8$, $a_6=a_m=13$, we have

$$\begin{aligned}(c+32)^n=&(c+30)^n+(c+28)^n-(c+24)^n-(c+18)^n+(c+16)^n-(c+10)^n\\&+(c+8)^n+(c+6)^n-(c+4)^n-(c-4)^n+(c-6)^n+(c-8)^n \quad ..(32),\\&-(c-10)^n+(c-16)^n-(c-18)^n-(c-24)^n+(c-28)^n+(c-30)^n\\&-(c-32)^n\end{aligned}$$

where $n=5$, 4, 3, 2, or 1.

Making c, in (32), $=7$, we have

$$39^n=37^n+35^n-31^n+15^n+13^n-9^n-21^n\ldots.(33),$$

where $n=5$, 3, or 1.

In (30), by making $c=a_1=a_2=\ldots.a_m=1$, we have

$$(m-1)^n=m(m-1)^n-\frac{m(m-1)}{2}(m-3)^n+\frac{m(m-1)(m-2)}{6}(m-5)^n-\ldots.$$

$$\ldots.-(-1)^m\frac{m(m-1)}{2}(5-m)^n+(-1)^m m(3-m)^n-(-1)^m(1-m)^n.(34).$$

Supposing m and n to be both even or both odd, (34) may be written

$$(m+1)^n=(m-1)^{n+1}-\frac{m}{2}(m-3)^{n+1}+\frac{m(m-1)}{6}(m-5)^{n+1}$$

$$-\frac{m(m-1)(m-2)}{24}(m-7)^{n+1}+\ldots(35).$$

In (35), when m is even, the root of the last term is 1, and when the former is odd the latter is 2. Thus:

Making $m=6$, we have

$$7^n=5^{n+1}-3.3^{n+1}+5.1^{n+1}\ldots.(36),$$

where $n=4$ or 2.

[To be Continued.]